

# WHEN EVERY PRINCIPAL IDEAL IS FLAT

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**ABSTRACT.** This paper deals with well-known notion of *PF*-rings, that is, rings in which principal ideals are flat. We give a new characterization of *PF*-rings. Also, we provide a necessary and sufficient condition for  $R \bowtie I$  (resp.,  $R/I$  when  $R$  is a Dedekind domain or  $I$  is a primary ideal) to be *PF*-ring. The article includes a brief discussion of the scope and precision of our results.

## 1. INTRODUCTION

All rings considered in this paper are assumed to be commutative with identity elements and all modules are unitary. We start by recalling some definitions.

A ring  $R$  is called a *PF*-ring if principal ideals of  $R$  are flat. Recall that  $R$  is a *PF*-ring if and only if  $R_Q$  is a domain for every prime (resp., maximal) ideal  $Q$  of  $R$ . For example, any domain and any semihereditary ring is a *PF*-ring (since a localization of a semihereditary ring by a prime (resp., maximal) ideal is a valuation domain). Note that a *PF*-ring is reduced by [12, Theorem 4.2.2, p. 114]. See for instance [12, 13].

An  $R$ -module  $M$  is called *P*-flat if, for any  $(s, x) \in R \times M$  such that  $sx = 0$ , then  $x \in (0 : s)M$ . If  $M$  is flat, then  $M$  is naturally *P*-flat. When  $R$  is a domain,  $M$  is *P*-flat if and only if it is torsion-free. When  $R$  is an arithmetical ring, then any *P*-flat module is flat (by [5, p. 236]). Also, every *P*-flat cyclic module is flat (by [5, Proposition 1(2)]). See for instance [5, 12].

The amalgamated duplication of a ring  $R$  along an ideal  $I$  is a ring that is defined as the following subring with unit element  $(1, 1)$  of  $R \times R$ :

$$R \bowtie I = \{(r, r + i) / r \in R, i \in I\}.$$

This construction has been studied, in the general case, and from the different point of view of pullbacks, by D'Anna and Fontana [8]. Also, in [7], they have considered the case of the amalgamated duplication of a ring, in not necessarily Noetherian setting, along a multiplicative canonical ideal in the sense of [14]. In [6] D'Anna has studied some properties of  $R \bowtie I$ , in

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order to construct reduced Gorenstein rings associated to Cohen-Macaulay rings and has applied this construction to curve singularities. On the other hand, Maimani and Yassemi, in [16], have studied the diameter and girth of the zero-divisor of the ring  $R \bowtie I$ . Some references are [7, 8, 9, 10, 16].

Let  $A$  and  $B$  be rings and let  $\varphi : A \rightarrow B$  be a ring homomorphism making  $B$  an  $A$ -module. We say that  $A$  is a module retract of  $B$  if there exists a ring homomorphism  $\psi : B \rightarrow A$  such that  $\psi \circ \varphi = id_A$ .  $\psi$  is called retraction of  $\varphi$ . See for instance [12].

Our first main result in this paper is Theorem 2.1 which gives us a new characterization of  $PF$ -rings. Also, we provide a necessary and sufficient condition for  $R \bowtie I$  (resp.,  $R/I$  when  $R$  is a Dedekind domain or  $I$  is a primary ideal) to be  $PF$ -ring. Our results generate new and original examples which enrich the current literature with new families of  $PF$ -rings with zero-divisors.

## 2. MAIN RESULTS

Recall that an  $R$ -module  $M$  is called  $P$ -flat if, for any  $(s, x) \in R \times M$  such that  $sx = 0$ , then  $x \in (0 : s)M$ . Now, we give a new characterization for a class of  $PF$ -rings, which is the first main result of this paper.

**Theorem 2.1.** *Let  $R$  be a commutative ring. Then the following conditions are equivalent:*

- (1) *Every ideal of  $R$  is  $P$ -flat.*
- (2) *Every principal ideal of  $R$  is  $P$ -flat.*
- (3)  *$R$  is a  $PF$ -ring, that is every principal ideal of  $R$  is flat.*
- (4) *For any elements  $(s, x) \in R^2$  such that  $sx = 0$ , there exists  $\alpha \in (0 : s)$  such that  $x = \alpha x$ .*

**Proof.** (1)  $\implies$  (2) Clear.

(2)  $\implies$  (3) Let  $Ra$  be a principal ideal of  $R$  generated by  $a$ . Our aim is to show that  $Ra$  is flat.

Let  $J$  be an ideal of  $R$ . We must show that  $u : Ra \otimes J \longrightarrow Ra \otimes R$ , where  $u(a \otimes x) = ax$ , is injective. Let  $a \in R$  and  $x \in J$  such that  $ax = 0$ . Hence, there exists  $\beta \in (0 : x)$  and  $\lambda \in R$  such that  $a = \beta\lambda a$  (since  $Ra$  is  $P$ -flat). Therefore,  $a \otimes x = \beta\lambda a \otimes x = \lambda a \otimes \beta x = 0$ , as desired.

(3)  $\implies$  (4) Let  $(s, x)$  be an element of  $R^2$  such that  $sx = 0$ . Our aim is to show that there exists  $\beta \in (0 : s)$  such that  $x = \beta x$ . The principal ideal generated by  $x$  is  $P$ -flat (since it is flat), so there exists  $\alpha \in (0 : s)$  and  $r \in R$  such that  $x = \alpha rx = \beta x$  with  $\beta = \alpha r \in (0 : s)$ .

(4)  $\implies$  (1) Let  $I$  be an ideal of  $R$ . Let  $(s, x) \in R \times I$  such that  $sx = 0$ .

Hence, there exists  $\alpha \in (0 : s)$  such that  $x = \alpha x$  and so  $x \in (0 : s)I$ . Therefore,  $I$  is  $P$ -flat, as desired.  $\square$

**Corollary 2.2.** *Let  $R$  be a ring. The following conditions are equivalent:*

- (1) *Every ideal of  $R$  is  $P$ -flat.*
- (2) *Every ideal of  $R_Q$  is  $P$ -flat for every prime ideal  $Q$  of  $R$ .*
- (3) *Every ideal of  $R_m$  is  $P$ -flat for every maximal ideal  $m$  of  $R$ .*
- (4)  *$R_Q$  is a domain for every prime ideal  $Q$  of  $R$ .*
- (5)  *$R_m$  is a domain for every maximal ideal  $m$  of  $R$ .*

**Proof.** By Theorem 2.1 and [12, Theorem 4.2.2].  $\square$

Recall that a ring  $R$  is called an arithmetical ring if the lattice formed by its ideals is distributive. If  $wgldim(R) \leq 1$ , then  $R$  is an arithmetical ring. See for instance [2, 3].

Now, we add a condition with arithmetical in order to have equivalence between arithmetical and  $wgldim(R) \leq 1$ .

**Proposition 2.3.** *Let  $R$  be a ring. Then the following conditions are equivalent:*

- (1)  $wgldim(R) \leq 1$ .
- (2)  $R$  is arithmetical and a  $PF$ -ring.
- (3)  $R$  is arithmetical and every principal ideal of  $R$  is flat.
- (4)  $R$  is arithmetical and every principal ideal of  $R$  is  $P$ -flat.
- (5)  $R$  is arithmetical and every ideal of  $R$  is  $P$ -flat.

**Proof.** 1)  $\Rightarrow$  2)  $\Rightarrow$  3)  $\Rightarrow$  4)  $\Rightarrow$  5). By Theorem 2.1.

5)  $\Rightarrow$  1). Assume that the ring  $R$  is arithmetical and every ideal of  $R$  is  $P$ -flat. Our aim is to show that  $wgldim(R) \leq 1$ . Let  $I$  be a finitely generated ideal of  $R$ . Hence,  $I$  is  $P$ -flat and so  $I$  is flat (since  $R$  is arithmetical by [5, p. 236]) and this completes the proof.  $\square$

Now we show that the localization of a  $PF$ -ring is always a  $PF$ -ring.

**Proposition 2.4.** *Let  $R$  be a  $PF$ -ring and let  $S$  be a multiplicative subset of  $R$ . Then  $S^{-1}(R)$  is a  $PF$ -ring.*

**Proof.** Assume that  $R$  is a  $PF$ -ring and let  $J$  be a principal ideal of  $S^{-1}(R)$ . We claim that  $J$  is flat. Indeed, since  $J$  is a principal ideal of  $S^{-1}R$ , then there exists an element  $\frac{a}{b}$  of  $J$  such that  $J = S^{-1}(R)\frac{a}{b}$ . Set  $I = Ra$ . Hence,  $I$  is flat since  $R$  is a  $PF$ -ring and so  $J(= S^{-1}(I))$  is a flat ideal of  $S^{-1}R$ . It follows that  $S^{-1}(R)$  is a  $PF$ -ring.  $\square$

Now, we study the transfer of  $PF$ -ring property to the direct product.

**Proposition 2.5.** *Let  $(R_i)_{i \in I}$  be a family of commutative rings. Then  $R = \prod_{i \in I} R_i$  is a  $PF$ -ring if and only if  $R_i$  is a  $PF$ -ring for all  $i \in I$ .*

**Proof.** Assume that  $R_i$  is a  $PF$ -ring for each  $i \in I$  and set  $R = \prod_{i \in I} R_i$ . Let  $x = (x_i)_{i \in I}$  and  $y = (y_i)_{i \in I}$  be two elements of  $R$  such that  $xy = 0$ . Then, for every  $i \in I$ , there exists  $\alpha_i \in (0 : x_i)$  such that  $y_i = \alpha_i x_i$  (since  $R_i$  is a  $PF$ -ring). Hence,  $(y_i)_{i \in I} = (\alpha_i)_{i \in I} (x_i)_{i \in I}$  and  $(\alpha_i)_{i \in I} (x_i)_{i \in I} = (\alpha_i x_i)_{i \in I} = 0$ . Therefore,  $R$  is a  $PF$ -ring.

Conversely, assume that  $R = \prod_{i \in I} R_i$  is a  $PF$ -ring and we claim that  $R_i$  is a  $PF$ -ring for every  $i \in I$ .

Indeed, let  $i \in I$  and let  $x_i, y_i$  be two elements of  $R_i$  such that  $x_i y_i = 0$ .

Consider  $x = (a_j)_{j \in I}$ , with  $\begin{cases} a_i = x_i, \\ a_j = 0 \text{ for } j \neq i. \end{cases}$  and  $y = (b_j)_{j \in I}$ , with

$\begin{cases} b_i = y_i, \\ b_j = 0 \text{ for } j \neq i. \end{cases}$  Since  $R$  is a  $PF$ -ring, then there exists  $\alpha \in (0 : x)$

such that  $y = \alpha x$  (that is, for all  $j \in I$ ,  $b_j = \alpha_j a_j$  and  $\alpha_j a_j = 0$ ). Hence,  $y_i = \alpha_i x_i$  with  $\alpha_i \in (0 : x_i)$ . Therefore,  $R_i$  is a  $PF$ -ring for all  $i \in I$  and this completes the proof.  $\square$

Next we study the transfer of  $PF$ -ring property to homomorphic image. First, the following example shows that the homomorphic image of a  $PF$ -ring is not always a  $PF$ -ring.

**Example 2.6.** *Let  $A$  be a domain and let  $R = A[X]$ . Then:*

- (1)  *$R$  is a  $PF$ -ring since it is a domain.*
- (2)  *$R/(X^n)$  (for  $n \geq 2$ ) is not a  $PF$ -ring since  $\overline{X^n} = 0$  and  $\overline{X} \neq 0$ .*

Recall that if  $R$  is a Dedekind domain and  $I$  is a nonzero ideal of  $R$ , then  $I = P_1^{\alpha_1} \dots P_n^{\alpha_n}$  for some distinct prime ideals  $P_1, \dots, P_n$  uniquely determined by  $I$  and some positive integers  $\alpha_1, \dots, \alpha_n$  uniquely determined by  $I$  (by [11, Theorem 3.14]).

Now, when  $R$  is a Dedekind domain or  $I$  is a primary ideal, we give a characterization of  $R$  and  $I$  such that  $R/I$  is a  $PF$ -ring.

**Theorem 2.7.** *Let  $R$  be a ring and let  $I$  be an ideal of  $R$ . Then:*

- (1) *Assume that  $R$  is a Dedekind domain and  $I = P_1^{\alpha_1} \dots P_n^{\alpha_n}$  a nonzero ideal of  $R$ , where  $P_1, \dots, P_n$  are the prime ideals defined by  $I$ . Then  $R/I$  is a  $PF$ -ring if and only if  $\alpha_i = 1$  for all  $i \in \{1, \dots, n\}$ .*

(2)  $I$  is a primary ideal of  $R$  and  $R/I$  is a  $PF$ -ring if and only if  $I$  is a prime ideal of  $R$ .

**Proof.** 1) Let  $R$  be a Dedekind domain and let  $I = P_1^{\alpha_1} \dots P_n^{\alpha_n}$  for  $P_1, \dots, P_n$  be nonzero prime ideals of  $R$ , then  $R/I = \prod_{i=1}^n (R/P_i^{\alpha_i})$ . Assume that  $\alpha_i = 1$  for all  $1 \leq i \leq n$ . Hence,  $R/P_i$  is a  $PF$ -ring since  $R/P_i$  is an integral domain, and so  $R/I = \prod_{i=1}^n (R/P_i^{\alpha_i})$  is a  $PF$ -ring by Proposition 2.5.

Conversely, assume that  $R/I = \prod_{i=1}^n (R/P_i^{\alpha_i})$  is a  $PF$ -ring. Let  $i \in \{1, \dots, n\}$ . Then  $R/P_i^{\alpha_i}$  is a  $PF$ -ring by Proposition 2.5. Hence,  $R/P_i^{\alpha_i}$  is reduced and so the intersection of all prime ideals  $Q$  of  $R/P_i^{\alpha_i}$  is zero (i.e.  $\bigcap_{Q \in \text{spect}(R/P_i^{\alpha_i})} Q = \{0\}$ ) by [1, Proposition 1.8]. On the other hand for every prime ideals  $Q$  of  $R/P_i^{\alpha_i}$ , there exist a prime ideal  $Q'$  of  $R$  such that  $P_i^{\alpha_i} \subset Q'$  and  $Q = Q'/P_i^{\alpha_i}$ , then,  $P_i/P_i^{\alpha_i} \subset Q$ . It follows that  $\{0\} = \bigcap_{Q \in \text{spect}(R/P_i^{\alpha_i})} Q = P_i/P_i^{\alpha_i}$  and so  $P_i = P_i^{\alpha_i}$ , since  $R$  is Dedekind domain then,  $\alpha_i = 1$ .

2) It's obvious that if  $I$  is a prime ideal, then  $R/I$  is a  $PF$ -ring and  $I$  is a primary ideal.

Conversely, assume that  $I$  is a primary ideal and  $R/I$  is a  $PF$ -ring. Our aim is to show that  $I$  is a prime ideal of  $R$ . Let  $x, y \in R$  such that  $xy \in I$ . We claim that  $x \in I$  or  $y \in I$ . Without loss of generality, we may assume that  $x \notin I$ . Since  $xy \in I$ , then there exists an integer  $n > 0$  such that  $y^n \in I$  (since  $I$  is a primary ideal). Hence,  $\overline{y}^n = 0$  and so  $\overline{y} = 0$  since  $R/I$  is a  $PF$ -ring; that is  $y \in I$ . Therefore,  $x \in I$  or  $y \in I$  and so  $I$  is a prime ideal of  $R$ , as desired.  $\square$

Now, we are able to give examples of  $PF$ -rings and non- $PF$ -rings.

**Example 2.8.** (1)  $\mathbb{Z}/4\mathbb{Z}$  is not a  $PF$ -ring by Theorem 2.8(1).

(2)  $\mathbb{Z}/30\mathbb{Z}$  is a  $PF$ -ring by Theorem 2.8(1).

Now, we study the transfer of a  $PF$ -property to amalgamated duplication of a ring  $R$  along an ideal  $I$ .

**Theorem 2.9.** Let  $R$  be a ring, and let  $I$  be an ideal of  $R$ . Then,  $R \bowtie I$  is a  $PF$ -ring if and only if  $R$  is a  $PF$  and  $I$  is pure.

We need the following lemma before proving this Theorem.

**Lemma 2.10.** *Let  $R$  and  $S$  be rings and let  $\varphi : R \rightarrow S$  be a ring homomorphism making  $R$  a module retract of  $S$ . If  $S$  is a PF-ring, then so is  $R$ .*

**Proof.** Let  $\varphi : R \rightarrow S$  be a ring homomorphism and let  $\psi : S \rightarrow R$  be a ring homomorphism such that  $\psi \circ \varphi = id_R$ . Let  $(x, y) \in R^2$  such that  $xy = 0$ . Then  $\varphi(x)\varphi(y) = \varphi(xy) = 0$ . Hence, there exists an element  $\alpha \in S$  such that  $\alpha\varphi(x) = 0$  and  $\varphi(y) = \alpha\varphi(y)$  (since  $S$  is a PF-ring) and so  $y = \psi(\varphi(y)) = \psi(\alpha\varphi(y)) = \psi(\alpha)y$  and  $\psi(\alpha)x = \psi(\alpha\varphi(x)) = \psi(0) = 0$ , as desired.  $\square$

**Proof.** of Theorem 2.9.

Assume that  $R \bowtie I$  is a PF-ring and we must to show that  $R$  is a PF-ring and  $I$  is a pure ideal of  $R$ . We can easily show that  $R$  is a module retract of  $R \bowtie I$  where the retraction map  $\varphi$  is defined by  $\varphi(r, r + i) = r$  and so  $R$  is a PF-ring by Lemma 2.10.

We claim that  $I_m \in \{0, R_m\}$  for every maximal ideal  $m$  of  $R$ . Let  $m$  be an arbitrary maximal ideal of  $R$ , we have:  $I \subseteq m$  or  $I \not\subseteq m$ . If  $I \not\subseteq m$  then,  $I_m = R_m$ . If  $I \subseteq m$ . Deny.  $I_m \notin \{0, R_m\}$  and so  $(R \bowtie I)_M = R_m \bowtie I_m$ , where  $M$  a maximal ideal of  $R \bowtie I$  such that  $M \cap R = m$ . Since  $R_m$  is a domain, then  $R_m \bowtie I_m$  is reduced and  $O_1 (= \{0\} \times I_m)$  and  $O_2 (= I_m \times \{0\})$  are the only minimal prime ideals of  $(R \bowtie I)_M$  by [8, Proposition 2.1]; hence it is not a PF-ring by [12, Theorem 4.2.2] (since  $(R \bowtie I)_M$  is local), a desired contradiction. Therefore,  $I_m \in \{0, R_m\}$  for every maximal ideal  $m$  of  $R$ .

Conversely, assume that  $R$  is a PF-ring and  $I$  is a pure ideal of  $R$ , i.e.  $I_m \in \{0, R_m\}$  for every maximal ideal  $m$  of  $R$ . Our aim is to prove that  $R \bowtie I$  is a PF-ring. Using Corollary 2.2, we need to prove that  $(R \bowtie I)_M$  is a PF-ring whenever  $M$  is a maximal ideal of  $R \bowtie I$ . Let  $M$  be an arbitrary maximal ideal of  $R \bowtie I$  and set  $m = M \cap R$ . Then, necessarily  $M \in \{M_1, M_2\}$ , where  $M_1 = \{(r, r + i)/r \in m, i \in I\}$  and  $M_2 = \{(r + i, r)/r \in m, r \in I\}$ , by [7, Theorem 3.5]. On the other hand,  $I_m \in \{0, R_m\}$ . Then, testing all cases of [6, Proposition 7], we have two cases:

(a)  $(R \bowtie I)_M \cong R_m$  if  $I_m = 0$  or  $I \not\subseteq m$ .

(b)  $(R \bowtie I)_M \cong R_m \times R_m$  if  $I_m = R_m$  and  $I \subseteq m$ .

Since  $R_m$  is a PF-ring (by Corollary 2.2), then so is  $R_m \times R_m$  by Proposition 2.5 and hence  $(R \bowtie I)_M$  is a PF-ring.  $\square$

**Corollary 2.11.** *Let  $R$  be a domain and let  $I$  be a proper ideal of  $R$ . Then  $R \bowtie I$  is never a PF-ring.*

**Corollary 2.12.** *Let  $(R, m)$  be a local ring and let  $I$  be a proper ideal of  $R$ . Then  $R \bowtie I$  is never a PF-ring.*

Now we are able to construct a class of  $PF$ -rings.

**Example 2.13.** *Let  $R$  be a  $PF$ -ring and let  $I = Re$ , where  $e$  is an idempotent element of  $R$ . Then  $R \bowtie I$  is a  $PF$ -ring by Theorem 2.9.*

The following example shows that a subring of  $PF$ -ring is not always a  $PF$ -ring. For any ring  $R$ , we denote by  $T(R)$  the total ring of quotients of  $R$ .

**Example 2.14.** *Let  $R$  be an integral domain,  $I$  a proper ideal of  $R$  and let  $S = R \bowtie I$ . Then:*

- (1)  *$S(= R \bowtie I)$  is not a  $PF$ -ring by Corollary 2.11.*
- (2)  *$R \bowtie I \subseteq R \times R$  and  $R \times R$  is a  $PF$ -ring by Proposition 2.5 (since  $R$  is a  $PF$ -ring).*
- (3)  *$T(S) = T(R \times R) = K \times K$ , where  $K = T(R)$ .*

We end this paper by showing that the transfer of  $PF$ -ring property to Pullback is not always a  $PF$ -ring.

**Example 2.15.** *Let  $R$  be a domain and  $I$  a proper ideal of  $R$ . Then:*

- (1) *The ring  $R \bowtie I$  can be obtained as a pullback of  $R$  and  $R \times R$  over  $R \times (R/I)$ .*
- (2) *The ring  $R \bowtie I$  is not a  $PF$ -ring by Corollary 2.11.*
- (2) *The rings  $R$  and  $R \times R$  are a  $PF$ -rings.*

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